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School on deformation theory — Roma 16/09/2024

§1. Functors of Artin rings

<u>Notation</u>: $\mathbb{k}$ field, $\mathcal{C}_{\mathbb{k}}$ category of local $\mathbb{k}$-algebras with residue field $\mathbb{k}$. The morphisms are local maps of $\mathbb{k}$-algebras.

<u>Exercise</u>: the set theoretical fiber product is also the fiber product in $\mathcal{C}_{\mathbb{k}}$.

<u>Definition</u>: $\mathrm{Art}_{\mathbb{k}} \subseteq \mathcal{C}_{\mathbb{k}}$ is the full subcategory of Artin local $\mathbb{k}$-algebras with residue field $\mathbb{k}$.

<u>Example</u>: $\mathbb{k}[t]/(t^{n+1}) \in \mathrm{Art}_{\mathbb{k}}$ $\forall n \in \mathbb{N}$.

<u>Exercise</u>:
- Prove that an algebra $A \in \mathcal{C}_{\mathbb{k}}$ belongs to $\mathrm{Art}_{\mathbb{k}}$ if and only if it is finite dimensional over $\mathbb{k}$.
- $\mathrm{Art}_{\mathbb{k}}$ is closed under fiber products.

<u>Definition</u>: A <u>small extension</u> e in $\mathrm{Art}_{\mathbb{k}}$ is an exact sequence of abelian groups $e: 0 \to M \xrightarrow{\alpha} B \xrightarrow{\beta} A \to 0$ s.t.

1. $\beta: B \to A$ is a morphism in $\mathrm{Art}_{\mathbb{k}}$
2. $\alpha: M \to B$ is a morphism of B-modules
3. $M \cdot m_B = 0$.

- The map β is called <u>small surjection</u>
- e is called <u>principal small extension</u> if $M = \mathbb{k}$.

<u>Rmk</u>: M is a finite dimensional vector space over $\mathbb{k}$.

<u>Exercise</u>: Prove that every surjective morphism in $\mathrm{Art}_{\mathbb{k}}$ is a finite composition of principal small surjections.

<u>Example</u>: $A \in \mathrm{Art}_{\mathbb{k}}$, M finite dimensional vector space over $\mathbb{k}$. The <u>trivial extension</u> is $0 \to M \to A \oplus M \to A \to 0$ where

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the product on $A \oplus M$ is defined by

$$(a_1, m_1) \cdot (a_2, m_2) = (a_1 \cdot a_2 \, , \, a_1 m_2 + a_2 m_1)$$

be careful and remember that M must be annihilated by the maximal ideal.

Definition: A morphism of small extensions $e_1 \xrightarrow{\varphi} e_2$ is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & M_1 & \to & B_1 & \to & A_1 & \to & 0 \\ & & \downarrow \varphi_M & & \downarrow \varphi_B & & \downarrow \varphi_A & & \\ 0 & \to & M_2 & \to & B_2 & \to & A_2 & \to & 0 \end{array}$$

where φ_B and φ_A are morphisms in $Art_{\mathbb{k}}$.

Definition: The pushout of a small extension e along a morphism $f: M \to N$ of $\mathbb{k}$-vector spaces is

$$\begin{array}{cccccccccc} e: & 0 & \to & M & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & A & \to & 0 \\ & & & f \downarrow & & \downarrow & & \downarrow id & & \\ f_* e: & 0 & \to & N & \to & \dfrac{B \oplus N}{\{(\alpha(m), -f(m))\}} & \to & A & \to & 0 \end{array}$$

Definition: A functor of Artin rings is a covariant functor $F: Art_{\mathbb{k}} \to Set$ such that $F(\mathbb{k}) = 0$.

Examples: ① $B \in Art_{\mathbb{k}}$ fixed. $h_B: Art_{\mathbb{k}} \to Set$, $A \longmapsto Hom_{Art_{\mathbb{k}}}(B, A)$

② $R \in \mathcal{C}_{\mathbb{k}}$ fixed. $h_R: Art_{\mathbb{k}} \to Set$, $A \longmapsto Hom_{\mathcal{C}_{\mathbb{k}}}(R, A)$

③ S Noetherian $\mathbb{k}$-algebra. M finitely generated S-module. An infinitesimal deformation of M over $A \in Art_{\mathbb{k}}$ is the data of:
- a $S \otimes A$-module M_A, flat over A
- a morphism $\pi: M_A \to M$ of $S \otimes A$-modules inducing an isomorphism $M_A \otimes_A \mathbb{k} \xrightarrow{\cong} M$

(here M is an $S \otimes A$-module via $A \to A/m_A = \mathbb{k}$).

two deformations M_A and M'_A of M over A are isomorphic if there exists a commutative diagram of A-modules

$$\begin{tikzcd} M_A \arrow[rr, "\cong"] \arrow[dr] & & M'_A \arrow[dl] \\ & M & \end{tikzcd}.$$

$$\mathrm{Def}_M : \mathrm{Art}_{\mathbb{k}} \to \mathrm{Set} \qquad A \mapsto \left\{ \begin{matrix} \text{deformations} \\ \text{of } M \text{ over } A \end{matrix} \right\} \Big/ \cong$$ is a functor of Artin rings.

④ S $\mathbb{k}$-algebra. An infinitesimal deformation of S over $A \in \mathrm{Art}_{\mathbb{k}}$ is a morphism $S_A \to S$ of A-algebras inducing an isomorphism $S_A \otimes_A \mathbb{k} \xrightarrow{\cong} S$, with S_A flat over A. (Here S is an A-algebra via $A \to \mathbb{k}$).

Two deformations $S_A \to S$ and $S'_A \to S$ are isomorphic if there exists a commutative diagram of A-algebras

$$\begin{tikzcd} S_A \arrow[rr, "\cong"] \arrow[dr] & \arrow[d, phantom, "\circlearrowright"] & S'_A \arrow[dl] \\ & S & \end{tikzcd}.$$

$$\mathrm{Def}_S : \mathrm{Art}_{\mathbb{k}} \to \mathrm{Set} \qquad A \mapsto \left\{ \begin{matrix} \text{deformations} \\ \text{of } S \text{ over } A \end{matrix} \right\} \Big/ \cong$$ is a functor of Artin rings.

Definition: $F : \mathrm{Art}_{\mathbb{k}} \to \mathrm{Set}$ functor of Artin rings.

$$\begin{tikzcd} B \times_A C \arrow[r] \arrow[d] & C \arrow[d] \\ B \arrow[r, "\beta"] & A \end{tikzcd} \rightsquigarrow \eta : F(B \times_A C) \to F(B) \times_{F(A)} F(C)$$

① F is a deformation functor if
- β surjective $\Rightarrow \eta$ surjective
- $A = \mathbb{k} \Rightarrow \eta$ bijective

② F is homogeneous if β surjective $\Rightarrow \eta$ bijective.

Remark: All the examples above are deformation functors.

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Definition: A natural transformation $F \to G$ between functors of Artin rings is <u>smooth</u> if for every surjection $B \xrightarrow{\beta} A$ in $\mathrm{Art}_{\mathbb{k}}$ the dotted morphism is surjective.

$$\begin{tikzcd}
F(B) \arrow[dr, dotted] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& G(B)\times_{G(A)} F(A) \arrow[r] \arrow[d] & F(A) \arrow[d] \\
& G(B) \arrow[r] & G(A)
\end{tikzcd}$$

Definition: A functor of Artin rings is smooth or <u>unobstructed</u> if the natural transformation $F \to 0$ is smooth.

Example: F is smooth if and only if

$\beta: B \to A$ surjective in $\mathrm{Art}_{\mathbb{k}} \Rightarrow F(B) \to F(A)$ surjective

Rmk: $h_R : \mathrm{Art}_{\mathbb{k}} \to \mathrm{Set}$ is smooth if and only if R is formally smooth.

Definition: Let F be a deformation functor. The <u>tangent space of F</u> is the vector space $T^1F = F\left(\mathbb{k}[\varepsilon]/(\varepsilon^2)\right)$.

Exercise: Prove that T^1F is a vector space.

Hint:
- $F\left(\mathbb{k}[\varepsilon] \times_{\mathbb{k}} \mathbb{k}[\varepsilon]\right) \xrightarrow{\cong} F(\mathbb{k}[\varepsilon]) \times_{F(\mathbb{k})} F(\mathbb{k}[\varepsilon]) = T^1F \times T^1F$
- use the map $\mathbb{k}[\varepsilon] \times_{\mathbb{k}} \mathbb{k}[\varepsilon] \xrightarrow{+} \mathbb{k}[\varepsilon]$
$$(a+b\varepsilon, a+b'\varepsilon) \mapsto a + (b+b')\varepsilon$$
- use the map $\mathbb{k}[\varepsilon] \xrightarrow{\lambda} \mathbb{k}[\varepsilon]$, $a+b\varepsilon \mapsto a+\lambda b\varepsilon$ to define the multiplication by $\lambda \in \mathbb{k}$.

Example: $T^1 h_R = \mathrm{Mor}_{C_{\mathbb{k}}}(R, \mathbb{k}[\varepsilon]) \cong \mathrm{Hom}_{\mathbb{k}}\left(\frac{m_R}{m_R^2}, \mathbb{k}\right)$

It is the Zariski tangent space of $\mathrm{Spec}(R)$ at its closed point.

Definition: A natural transformation between deformation functors is a <u>weak equivalence</u> if it is smooth and bijective on tangent spaces.

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<u>Definition</u>: A <u>hull</u> of a deformation functor F is a weak equivalence $h_R \longrightarrow F$

<u>theorem</u> [Schlessinger] Let F be a deformation functor such that $\dim_{\mathbb{k}} T^1F < \infty$. Then there exists a local complete Noetherian $\mathbb{k}$-algebra R with residue field $\mathbb{k}$ and a hull $h_R \longrightarrow F$.

§2. Obstructions

<u>Definition</u>: Let F be a functor of Artin rings. An <u>obstruction theory</u> for F with values in a $\mathbb{k}$-vector space V is the following data:

① for every small extension in $\mathrm{Art}_{\mathbb{k}}$

$$e: \ 0 \to M \to B \to A \to 0$$

a morphism $v_e: F(A) \to V \otimes_{\mathbb{k}} M$ satisfying

② the base change property

$$\begin{tikzcd}
e_1: 0 \arrow[r] & M_1 \arrow[r, "\alpha_1"] \arrow[d, "f"] & B_1 \arrow[r, "\beta_1"] \arrow[d, "g"] & A_1 \arrow[r] \arrow[d, "h"] & 0 \\
e_2: 0 \arrow[r] & M_2 \arrow[r, "\alpha_2"'] & B_2 \arrow[r, "\beta_2"'] & A_2 \arrow[r] & 0
\end{tikzcd}
\quad \Longrightarrow \quad
\begin{tikzcd}
F(A_1) \arrow[r, "v_{e_1}"] \arrow[d, "F(h)"'] \arrow[dr, phantom, "\circlearrowleft"] & V \otimes M_1 \arrow[d, "id \otimes f"] \\
F(A_2) \arrow[r, "v_{e_2}"'] & V \otimes M_2
\end{tikzcd}$$

<u>Proposition</u>: F functor of Artin rings. (V, v) obstruction theory for F. $e: 0 \to M \to B \xrightarrow{\beta} A \to 0$ small extension in $\mathrm{Art}_{\mathbb{k}}$. Assume $a \in F(A)$ lifts to $F(B)$ [i.e. $\exists b \in F(B)$ s.t. $F(\beta)(b) = a$]. Then $v_e(a) = 0$.

<u>Proof</u>: Consider the morphism of small extensions

$$\begin{tikzcd}
e': 0 \arrow[r] & 0 \arrow[r] \arrow[d] & B \arrow[r, "id"] \arrow[d, "id"] & B \arrow[r] \arrow[d, "\beta"] & 0 \\
e: 0 \arrow[r] & M \arrow[r, "\alpha"'] & B \arrow[r, "\beta"'] & A \arrow[r] & 0
\end{tikzcd}
\quad \Longrightarrow \quad
\begin{tikzcd}
F(B) \arrow[r, "v_{e'}"] \arrow[d, "F(\beta)"'] \arrow[dr, phantom, "\circlearrowleft"] & V \otimes 0 \arrow[d] \\
F(A) \arrow[r, "v_e"'] & V \otimes M
\end{tikzcd}$$

$\Longrightarrow v_e \circ F(\beta) \equiv 0$. $\square$

Definition: A functor of Artin rings admits a complete obstruction theory if there exists (V,v) obstruction theory satisfying the converse of the Proposition:

$$a\in F(A) \text{ lifts to } F(B) \iff v_e(a)=0.$$

Remark: A functor of Artin rings is smooth if and only if the trivial obstruction theory $(0,0)$ is complete.

Prop: Every obstruction theory is determined by the values on principal small extensions.

Pf: $e:\ 0\to M\to B\to A\to 0$ small extension in $\mathrm{Art}_{\mathbb{K}}$

$a\in F(A) \rightsquigarrow v_e(a)\in V\otimes M$ is uniquely determined by the values $(Id_V\otimes f)v_e(a)\in V$, where f varies along a basis of $\mathrm{Hom}_{\mathbb{K}}(M,\mathbb{K})$.

[this is the same as fixing a basis for M and looking at the coordinates of $v_e(a)\in V\otimes M$.]

By base change:

$$\begin{tikzcd}
e: 0 \arrow[r] & M \arrow[r, "\alpha"] \arrow[d, "f"'] & B \arrow[r, "\beta"] \arrow[d, "{(id,0)}"] & A \arrow[r] \arrow[d, equal] & 0 \\
f_*e: 0 \arrow[r] & \mathbb{K} \arrow[r, "{(0,id)}"] & \dfrac{B\oplus\mathbb{K}}{\{(\alpha(m),-f(m))\}} \arrow[r, "\beta"] & A \arrow[r] & 0
\end{tikzcd}
\Rightarrow
\begin{tikzcd}
F(A) \arrow[r, "v_e"] \arrow[d, "id"'] \arrow[dr, phantom, "\circlearrowleft"] & V\otimes M \arrow[d, "id\otimes f"] \\
F(A) \arrow[r, "v_{f_*e}"'] & V\otimes\mathbb{K}
\end{tikzcd}$$

$\Rightarrow v_{f_*e}(a) = (id\otimes f)\,v_e(a).$ □

Thm: [Standard smoothness criterion]

Let $\eta: F\to G$ be a natural transformation of deformation functors, and let (V,v) and (W,w) be complete obstruction theories for F and G respectively. Assume that:

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① the map $\eta : T^1F \to T^1G$ is surjective

② there exists an injective linear map $V \xrightarrow{f} W$ such that for every principal small extension $e: 0 \to \mathbb{K} \to B \xrightarrow{\beta} A \to 0$

$$\begin{array}{ccc} F(A) & \xrightarrow{v_e} & V \\ \eta_A \downarrow & \circlearrowright & \downarrow f \\ G(A) & \xrightarrow[w_e]{} & W \end{array}$$

Then η is smooth.

Definition: • A <u>morphism of obstruction theories</u> $(V,v) \to (W,w)$ is a morphism of $\mathbb{K}$-vector spaces $f: V \to W$ such that

$$\begin{array}{ccc} F(A) & \xrightarrow{v_e} & V \otimes M \\ \| & \circlearrowright & \downarrow f \otimes id_M \\ F(A) & \xrightarrow{w_e} & W \otimes M \end{array} \qquad \forall e: 0 \to M \to B \to A \to 0$$

• An obstruction theory (O_F, ob) is called <u>universal</u> if for every obstruction theory (W,w) there exists a unique morphism $(O_F, ob) \to (W,w)$.

O_F is called the <u>obstruction space</u> of F.

<u>theorem</u>: Let F be a deformation functor. Then:

① there exists a (unique) universal obstruction theory for F, which is complete

② every element of the obstruction space O_F is of the form $ob_e(a)$ for some principal small extension $e: 0 \to \mathbb{k} \to B \to A \to 0$ and some $a \in F(A)$.

<u>Remark</u>: A deformation functor is smooth if and only if the trivial obstruction theory is universal.

[the trivial obstruction theory is $(0,0)$.]

Corollary: Let (V, v) be a complete obstruction theory for a deformation functor F. Then the obstruction space O_F is isomorphic to the vector subspace of V generated by all the obstructions arising from principal extensions.

Pf: Denote by $\theta: O_F \to V$ the universal map. Every principal obstruction is contained in the image of θ. Moreover, the morphism θ is injective, being (V, v) complete.

In fact, given $0 \to \mathbb{k} \to B \xrightarrow{\beta} A \to 0$ we have

$$\begin{tikzcd} F(A) \arrow[r, "ob_e"] \arrow[d, "id"'] & O_F \arrow[d, "\theta"] \\ F(A) \arrow[r, "v_e"] & V \end{tikzcd}$$

, so that if $\theta(c) = 0$ then there exists $a \in F(A)$ such that $\begin{cases} ob_e(a) = c \\ v_e(a) = 0 \end{cases}$

Now, (V, v) is complete $\Rightarrow$ a lifts to $F(B)$

$\Rightarrow ob_e(a) = 0 \Rightarrow c = 0 \Rightarrow \theta$ injective

Example: Let $R = P/I$ where $P = \mathbb{k}[[x_1, \dots, x_n]]$ and $I \subseteq m_P^2$

[Notice that any local complete Notherian $\mathbb{k}$-algebra with residue field $\mathbb{k}$ and embedding dimension n can be realized like this] Then $T^2 h_R := \mathrm{Hom}_P(I, \mathbb{k}) = \mathrm{Hom}_{\mathbb{k}}(\frac{I}{m_P I}, \mathbb{k})$ is the obstruction space for h_R.

Pf: For every $e: 0 \to M \xrightarrow{\alpha} B \xrightarrow{\beta} A \to 0$ and every $f \in h_R(A)$ we construct lifts:

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r] \arrow[d, dotted, "g|_I"] & P \arrow[r, "\pi"] \arrow[d, dotted, "g"] & R = P/I \arrow[r] \arrow[d, "f"] & 0 \\ 0 \arrow[r] & M \arrow[r, "\alpha"'] & B \arrow[r, "\beta"'] & A \arrow[r] & 0 \end{tikzcd}$$

Now, $m_B M = 0 \Rightarrow g(m_P I) \subseteq m_B M = 0$

$\rightsquigarrow \quad h_R(A) \xrightarrow{v_e} T^2 h_R \otimes M$

$\qquad f \longmapsto g|_I \in \mathrm{Hom}_{\mathbb{k}}(I/m_P I, M)$

Notice that v_e does not depend on the choice of g.

If $\tilde{g}: P \to B$ is another lifting of $P \xrightarrow{\pi} P/I \xrightarrow{f} A$, then

- $(g-\tilde{g})(m_P) \subseteq \ker\beta = \alpha(M)$
- $(g-\tilde{g})(I) \subseteq (g-\tilde{g})(m_P^2) \subseteq m_B \cdot \alpha(M) = 0$

$\Rightarrow T^2 h_R$ is an obstruction theory for h_R.

Our aim is now to prove that it is complete.

Assume $v_e(f)=0$. Then $g(I)=0$. $\Rightarrow \exists\, l: R \to B$ such that

$$\begin{tikzcd} P \arrow[r, "\pi"] \arrow[d, "g"'] & R = P/I \arrow[dl, "l"] \arrow[dl, phantom, "\circlearrowright", near start, shift right=3] \\ B & \end{tikzcd}$$

Now $f \circ \pi = \beta \circ g = \beta \circ l \circ \pi$ and we obtain $f = \beta \circ l$ by the surjectivity of π.

$$\Rightarrow \quad h_R(B) \xrightarrow{\beta\circ -} h_R(A) \quad \Rightarrow f \text{ lifts to } h_R(B).$$
$$l \longmapsto f$$

$\Rightarrow T^2 h_R$ is a complete obstruction theory for h_R.

<u>Remark</u>: Given a natural transformation of deformation functors $\phi: F \to G$, then $(O_G, ob^G \circ \phi)$ is an obstruction theory for F.

$\Rightarrow \exists!\, ob_\phi : O_F \to O_G$ compatible with ϕ:

$$\begin{tikzcd} F(A) \arrow[r, "ob_e^F"] \arrow[d, "\phi_A"'] & O_F \otimes M \arrow[d, "ob_\phi \otimes id_M"] \\ G(A) \arrow[r, "ob_e^G"] & O_G \otimes M \end{tikzcd} \qquad \forall e: 0 \to M \to B \to A \to 0$$

(*) <u>Remark</u>: If $\phi: F \to G$ is a natural transformation of deformation functors and G is smooth [i.e. $O_G = 0$] then every map of complete obstruction theories $f:(V,v) \to (W,w)$ for F and G respectively fits in a commutative diagram:

$$\begin{tikzcd} F(A) \arrow[r] \arrow[d, "\phi_A"'] & O_F \otimes M \arrow[r, hook] \arrow[d, "ob_\phi \otimes id_M"] & V \otimes M \arrow[d, "f \otimes id_M"] \\ G(A) \arrow[r] & \underset{O_G \otimes M}{\underset{\parallel}{0}} \arrow[r, hook] & W \otimes M \end{tikzcd}$$

Therefore every obstruction in $V \otimes M$ is annihilated by $f \otimes id_M$

<u>thm</u>: $\phi : F \to G$ natural transformation of deformation functors.

Then the following are equivalent:

① ϕ is smooth

② $T^1\phi : T^1F \to T^1G$ surjective and $ob_\phi : O_F \to O_G$ bijective

③ $T^1\phi : T^1F \to T^1G$ surjective and $ob_\phi : O_F \to O_G$ injective

<u>Pf</u>: ① ⇒ ② if $ob_\phi(z) = 0 \Rightarrow z = ob_e^F(a)$ for some $a \in F(A)$ and some principal small extension e

$$\begin{tikzcd}
F(A) \arrow[r, "ob_e^F"] \arrow[d, "\phi_A"'] & O_F \otimes k \arrow[d, "ob_\phi"] \\
G(A) \arrow[r, "ob_e^G"'] & O_G \otimes k
\end{tikzcd}
\qquad a \mapsto z \mapsto 0$$

then by base change $ob_e^G(\phi(a)) = 0$

$\Rightarrow \phi_A(a)$ lifts to $G(B)$.

$$\begin{tikzcd}
F(B) \arrow[rr, bend left] \arrow[dr] \arrow[ddr, bend right] & & \\
& G(B) \times_{G(A)} F(A) \arrow[r] \arrow[d] & F(A) \arrow[d, "\phi_A"] \\
& G(B) \arrow[r] & G(A)
\end{tikzcd}$$

$b' \mapsto (b,a)$, $\quad a \mapsto \phi_A(a)$, $\quad b \mapsto \phi_A(a)$

By the smoothness of ϕ, (b,a) lifts to $b' \in F(B)$

and then $\begin{matrix} F(B) \to F(A) \\ b' \mapsto a \end{matrix} \Rightarrow z = ob_e^F(a) = 0$.

② ⇒ ③ is clear.

③ ⇒ ① follows from the standard smoothness criterion.

<u>Corollary</u>: $\phi : F \to G$ natural transformation of deformation functors

Then the following are equivalent:

① ϕ is a weak equivalence

② $T^1\phi : T^1F \to T^2F$ and $ob_\phi : O_F \to O_G$ are bijective.

§3. Deformations of vector bundles

X smooth variety over $\mathbb{K}$. $\mathcal{E}$ locally free sheaf on X,
$\mathbb{K}$ fixed field of char $\mathbb{K}=0$.
Let $\mathcal{U}=\{U_i\}_{i\in I}$ be an affine open cover such that $\mathcal{E}|_{U_i}$ is free $\forall i\in I$.

Aim: Study the deformation functor

$$\mathrm{Def}_{\mathcal{E}}: \mathrm{Art}_{\mathbb{K}} \longrightarrow \mathrm{Set}$$
$$(A,m_A) \longmapsto \left\{\begin{array}{l}\text{locally free sheaves } \mathcal{E}_A \text{ over } X\times \mathrm{Spec}A \\ \text{flat over } A \text{ with a morphism } \mathcal{E}_A\to\mathcal{E} \\ \text{inducing an isomorphism } \mathcal{E}_A\otimes_A \mathbb{K}\to\mathcal{E}\end{array}\right\}/\cong$$

Lemma: There exists a natural isomorphism of groups
$$\exp(\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E},\mathcal{E})\otimes m_A)\cong\{\text{automorphisms of the trivial def } \mathcal{E}\otimes A\}$$

Pf: An automorphism of the trivial deformation $\mathcal{E}\otimes A$ is $g\in \mathrm{Hom}_{\mathcal{O}_X\otimes A}(\mathcal{E}\otimes A,\mathcal{E}\otimes A)$ such that its reduction
$$g\otimes_A \mathrm{Id}_{\mathbb{K}}: \mathcal{E}\otimes_{\mathbb{K}} A\otimes_A \mathbb{K}\longrightarrow \mathcal{E}\otimes_{\mathbb{K}} A\otimes_A\mathbb{K}$$
is the identity. This is equivalent to require that $\mathrm{im}(g-\mathrm{id})\subset\mathcal{E}\otimes m_A$, and
$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E},\mathcal{E})\otimes m_A=\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}\otimes m_A)=\mathrm{Hom}_{\mathcal{O}_X\otimes A}(\mathcal{E}\otimes A,\mathcal{E}\otimes m_A)$$
so that we have an isomorphism of Lie algebras
$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E},\mathcal{E})\otimes m_A\cong\{f\in\mathrm{Hom}_{\mathcal{O}_X\otimes A}(\mathcal{E}\otimes A,\mathcal{E}\otimes A)\mid \mathrm{im}(f)\subseteq\mathcal{E}\otimes m_A\}.$$

- Now, since f is a nilpotent endomorphism, e^f is defined and $\mathrm{im}(e^f-\mathrm{id})\subseteq\mathcal{E}\otimes m_A$.
- Conversely, if $g\in\mathrm{Hom}_{\mathcal{O}_X\otimes A}(\mathcal{E}\otimes A,\mathcal{E}\otimes A)$ is an automorphism of the trivial deformation, then define $f:=\log(\mathrm{id}+(g-\mathrm{id}))$ to obtain $g=e^f$. □

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Exercise: Prove that every local deformation of $\mathcal{E}|_{U_i}$ is trivial. In other words: if X is affine, then a vector bundle admits only the trivial deformation. [Hint after Corollary]

theorem: X smooth variety, $\mathcal{E}$ vector bundle. Consider the sheaf of Lie algebras $\mathcal{L} := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E})$, and the associated deformation functor $\mathrm{Def}_{\mathcal{L},\mathcal{U}} : \mathrm{Art}_{\mathbb{k}} \longrightarrow \mathrm{Set}$ [see Nathan's lecture notes] where $\mathcal{U}$ is an affine open cover such that $\mathcal{E}|_{U_i}$ is free. Then there exists a natural isomorphism of functors $\mathrm{Def}_{\mathcal{E}} \cong \mathrm{Def}_{\mathcal{L},\mathcal{U}}$.

Pf: Fix $A \in \mathrm{Art}_{\mathbb{k}}$. For every $i \in I$ we have isomorphisms

$$\varphi_i : \mathcal{E}|_{U_i} \otimes A \xrightarrow{\cong} \mathcal{E}_A|_{U_i} \quad [\text{here } \mathcal{E}_A \in \mathrm{Def}_{\mathcal{E}}(A) \text{ is fixed}].$$

- Then define $g_{ij} := \varphi_i^{-1} \circ \varphi_j|_{U_{ij}} : \mathcal{E}|_{U_{ij}} \otimes A \xrightarrow{\cong} \mathcal{E}|_{U_{ij}} \otimes A$ and notice that $g_{ij} \otimes_A \mathbb{k} = \mathrm{id}$. Moreover:

$$g_{jk} \circ g_{ik}^{-1} \circ g_{ij} = \mathrm{Id} \quad \text{on } \mathcal{E}|_{U_{ijk}} \otimes A.$$

- What if we choose another collection of isomorphisms $\varphi_i' : \mathcal{E}|_{U_i} \otimes A \xrightarrow{\cong} \mathcal{E}_A|_{U_i}$.

Then define $\sigma_i := \varphi_i^{-1} \circ \varphi_i' : \mathcal{E}|_{U_i} \otimes A \xrightarrow{\cong} \mathcal{E}|_{U_i} \otimes A$

$\Longrightarrow \varphi_i' = \varphi_i \circ \sigma_i \Longrightarrow g_{ij}' = \sigma_i^{-1} \circ \varphi_i^{-1} \circ \varphi_j \circ \sigma_j = \sigma_i^{-1} \circ g_{ij} \circ \sigma_j$.

- We have proven that there is a map

$$\mathrm{Def}_{\mathcal{E}}(A) \longrightarrow \mathrm{Def}_{\mathcal{L},\mathcal{U}}$$

$$\mathcal{E}_A \longmapsto \left\{ \{g_{ij}\} \in \mathrm{Aut}_{\mathcal{O}_X \otimes A}(\mathcal{E} \otimes A) \;\middle|\; \begin{matrix} g_{ij} \otimes_A \mathbb{k} = \mathrm{id} \\ g_{jk} \circ g_{ik}^{-1} \circ g_{ij} = \mathrm{id} \end{matrix} \right\} \Big/ \sim$$

$$\text{Lemma} \; \Vert$$

$$\left\{ \{f_{ij}\} \in \mathrm{Hom}(\mathcal{E}_{ij}, \mathcal{E}_{ij}) \otimes \mathfrak{m}_A \;\middle|\; \diamond(\{f_{ij}\}) = 0 \right\} \Big/ \sim$$

where $\diamond(\{f_{ij}\})_{rst} = f_{st} \star - f_{rt} \star f_{rs}$ [$\star$ is Baker-Campbell-Hausdorff]

- The map above is a bijection.

Rmk: We have seen in Nathan's lectures that:
if $\mathcal{L}$ is a sheaf of Lie algebras and $\mathcal{U}=\{U_i\}$ is an open ~~affine~~ cover
then $\cdot\ T^1\mathrm{Def}_{\mathcal{L},\mathcal{U}} = \check{H}^1(\mathcal{U},\mathcal{L})$

$\cdot\ (\check{H}^2(\mathcal{U},\mathcal{L}), \diamond)$ provides a complete obstruction theory for $\mathrm{Def}_{\mathcal{L},\mathcal{U}}$.

Corollary: $\mathcal{E}$ locally free sheaf on X. Then:

1. $T^1\mathrm{Def}_{\mathcal{E}} = \mathrm{Ext}^1(\mathcal{E},\mathcal{E})$
2. there exists a complete obstruction theory for $\mathrm{Def}_{\mathcal{E}}$ with values in the kernel of the trace map
$$\mathrm{Tr}: \mathrm{Ext}^2(\mathcal{E},\mathcal{E}) = H^2(X,\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E})) \longrightarrow H^2(X,\mathcal{O}_X)$$

Pf: 1. $T^1\mathrm{Def}_{\mathcal{E}} = T^1\mathrm{Def}_{\mathcal{L},\mathcal{U}} = \check{H}^1(\mathcal{U},\mathcal{L}) = \check{H}^1(\mathcal{U},\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}))$
$= H^1(X,\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E})) = \mathrm{Ext}^1(\mathcal{E},\mathcal{E})$.

2. $(\check{H}^2(\mathcal{U},\mathcal{L}),\diamond) = (\mathrm{Ext}^2(\mathcal{E},\mathcal{E}),\diamond)$ is a complete obstruction theory for $\mathrm{Def}_{\mathcal{L},\mathcal{U}}$. We can do better.

- The trace map $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}) \longrightarrow \mathcal{O}_X$ induces a natural transformation of deformation functors
$$\mathrm{Def}_{\mathcal{L},\mathcal{U}} \longrightarrow \mathrm{Def}_{\mathcal{O}_X,\mathcal{U}}$$
[$\mathcal{O}_X$ is a sheaf of Lie algebras with trivial $[-,-]$]
and a morphism between their complete obstruction theories
$$\mathrm{Tr}: \mathrm{Ext}^2(\mathcal{E},\mathcal{E}) \longrightarrow H^2(X,\mathcal{O}_X)$$
- By Remark ⊛ we have a commutative diagram
$$\begin{tikzcd}
\mathrm{Def}_{\mathcal{E}}(A) \arrow[r] \arrow[d] & O_{\mathcal{E}} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{Ext}^2(\mathcal{E},\mathcal{E}) \arrow[d, "\mathrm{Tr}"] \\
\mathrm{Def}_{\mathcal{O}_X,\mathcal{U}}(A) \arrow[r] & 0 \arrow[r] & H^2(X,\mathcal{E})
\end{tikzcd}$$
□

Ex: prove that $\mathrm{Def}_{\mathcal{O}_X,\mathcal{U}}$ is smooth

<u>Exercise</u>: If X is affine, then the vector bundle $\mathcal{E}$ (of finite rank) only admits the trivial deformation.

<u>Hint</u>: Consider a principal small extension $0 \to \mathbb{k} \to B \to A \to 0$ in $\mathrm{Art}_{\mathbb{k}}$ and tensor with the flat B-module $\mathcal{E}_B$ (which is a given deformation over B). Then

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}_B \longrightarrow \mathcal{E}_B \otimes_B A \to 0$$

is an exact sequence and by induction on the length of B we may assume that $\mathcal{E}_B \otimes_B A \cong \mathcal{E} \otimes_{\mathbb{k}} A$. Now show that this isomorphism can be lifted to an isomorphism $\mathcal{E}_B \cong \mathcal{E} \otimes_{\mathbb{k}} B$.

§4. Deformations of coherent sheaves.

X smooth projective variety, $\mathcal{F}$ coherent sheaf of $\mathcal{O}_X$-modules.

Recall:

$$\mathrm{Def}_{\mathcal{F}} : \mathrm{Art}_{\mathbb{k}} \longrightarrow \mathrm{Set}$$

$$A \longmapsto \left\{ \begin{array}{l} \text{coherent sheaf } \mathcal{F}_A \text{ of } \mathcal{O}_X \otimes A\text{-modules} \\ \text{on } X \times \mathrm{Spec} A, \text{ flat over } A, \text{ with a map} \\ \text{of sheaves } \mathcal{F}_A \to \mathcal{F} \text{ inducing } \mathcal{F}_A \otimes_A \mathbb{k} \cong \mathcal{F} \end{array} \right\} \Big/ \cong$$

$\mathcal{F}_A$ and $\mathcal{F}_A'$ are isomorphic if there is a commutative diagram

$$\begin{tikzcd} \mathcal{F}_A \arrow[rr, "\cong"] \arrow[dr] & & \mathcal{F}_A' \arrow[dl] \\ & \mathcal{F} & \end{tikzcd}$$

<u>Affine case</u>: if $X = \mathrm{Spec}(R)$ then we are deforming a R-module M. Considere a free resolution of R-modules

$$\cdots \xrightarrow{d} R^{\oplus n_1} \xrightarrow{d} R^{\oplus n_0} \xrightarrow{d} M \to 0$$

Given a deformation M_A of M, we can lift the above exact sequence to an exact sequence

$$\cdots \xrightarrow{d_A} R^{\oplus n_1} \otimes_{\mathbb{k}} A \xrightarrow{d_A} R^{\oplus n_0} \otimes_{\mathbb{k}} A \xrightarrow{d_A} M_A \to 0$$

that reduces to the first one when tensored by $- \otimes_A \mathbb{k}$

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Exercise: prove that such lift exists.

Conversely, given such an exact sequence of $R\otimes A$ modules, the module M_A is necessarily flat over A.

Exercise: Prove the statement above.

To conclude, notice that an isomorphism of deformations $M_A \to M_A'$ lifts to an isomorphism between the corresponding resolutions.

Morally, deformations of M are the same as deformations of the complex $(\overset{n_i}{R}\otimes A, d_A)$.

How to deal with the global case?

Consider a locally free resolution

$$0 \to \mathcal{E}^{-m} \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{E}^{-1} \xrightarrow{d} \mathcal{E}^{0} \xrightarrow{d} \mathcal{F} \to 0.$$

Denote by $(\mathcal{E}^\cdot, d)$ the complex

$$0 \to \mathcal{E}^{-m} \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{E}^{-1} \xrightarrow{d} \mathcal{E}^{0} \to 0$$

so that we have a quasi isomorphism of complexes of $\mathcal{O}_X$-modules $(\mathcal{E}^\cdot, d) \to (\mathcal{F}, 0)$ [here $\mathcal{F}$ is a complex concentrated in degree 0]

Fix a covering $\mathcal{U} = \{U_i\}_{i\in I}$ such that $\mathcal{E}^k|_{U_i}$ is free for every $k \leq 0$ and every $i \in I$.

- Following the idea of the affine case, for every $i\in I$ we can consider a deformation of $(\mathcal{E}^\cdot|_{U_i}, d)$ as a complex $(\mathcal{E}^\cdot|_{U_i}\otimes A, d + l_i)$, where $\{l_i\} \in \prod_{i\in I} End^1(\mathcal{E}^\cdot)(U_i)\otimes m_A$.

Notice that the deformed complex is a complex if and only if $0 = (d+l_i)^2 = dl_i + l_i d + l_i^2 = [d, l_i] + \frac{1}{2}[l_i, l_i]$ for all $i \in I$.

This is the Maurer-Cartan equation.

- Note that, by upper semicontinuity of cohomology, the complex $(\mathcal{E}^\cdot|_{U_i} \otimes A, d + l_i)$ can only have non trivial cohomology in degree 0.
- <u>Gluing</u>: To glue the deformed complexes we need isomorphisms on double intersections, such that they lift the identity. As yesterday, they can be written as exponentials:

$$e^{m_{ij}} : (\mathcal{E}^\cdot|_{U_{ij}} \otimes A, d + l_j) \longrightarrow (\mathcal{E}^\cdot|_{U_{ij}} \otimes A, d + l_i)$$

with $\{m_{ij}\} \in \prod_{i<j} End^0(\mathcal{E}^\cdot)(U_{ij}) \otimes m_A$.

Let us understand the compatibility with differentials:

$$\begin{tikzcd} \mathcal{E}^\cdot|_{U_{ij}} \otimes A \arrow[r, "e^{m_{ij}}"] \arrow[d, "d + l_j|_{U_{ij}}"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{E}^\cdot|_{U_{ij}} \otimes A \arrow[d, "d + l_i|_{U_{ij}}"] \\ \mathcal{E}^\cdot|_{U_{ij}} \otimes A \arrow[r, "e^{m_{ij}}"'] & \mathcal{E}^\cdot|_{U_{ij}} \otimes A \end{tikzcd}$$

that is:
$$d + l_i|_{U_{ij}} = e^{m_{ij}} (d + l_j|_{U_{ij}}) e^{-m_{ij}}$$

or equivalently:
$$l_i|_{U_{ij}} = e^{m_{ij}} * l_j|_{U_{ij}}$$

gauge action

- <u>Cocycle condition</u>: A key point is that we do not want to glue together the deformed complexes, but their cohomologies instead. Hence we only require the cocycle condition "up to homotopy" on the triple intersections. Passing to the logarithm we obtain

$$m_{jk} * -m_{ik} * m_{ij} = [d + l_j|_{U_{ijk}}, n_{ijk}]$$

BCH product

for some $\{n_{ijk}\} \in \prod_{i<j<k} End^{-1}(\mathcal{E}^\cdot)(U_{ijk}) \otimes m_A$.

<u>Remark:</u> · The collection of A-flat sheaves of $\mathcal{O}_X \otimes A$-modules

$$\mathcal{F}_{A,U_i} := \mathcal{H}^*(\mathcal{E}^\cdot|_{U_i} \otimes A, d+l_i) = \mathcal{H}^0(\mathcal{E}^\cdot|_{U_i} \otimes A, d+l_i)$$

glue to a coherent sheaf $\mathcal{F}_A$ which gives the desired deformation of $\mathcal{F}$.

Conversely, given a deformation $\mathcal{F}_A$ of $\mathcal{F}$ gives rise to a deformed local complex $(\mathcal{E}^\cdot|_{U_i} \otimes A, d+l_i)$ whose 0-th cohomology is $\mathcal{F}_A|_{U_i}$. These deformed resolutions are linked via isomorphisms lifting the identity on $\mathcal{F}_A$, and they satisfy the cocycle condition up to homotopy since liftings are unique up to homotopy.

· <u>Isomorphic deformations:</u> If $\mathcal{F}_A$ and $\mathcal{F}_A'$ are isomorphic deformations of $\mathcal{F}$, they correspond to the deformation data (l, m) and (l', m') respectively. The isomorphism $\mathcal{F}_A \cong \mathcal{F}_A'$ lifts to local isomorphisms between the deformed complexes: $e^{a_i}: (\mathcal{E}^\cdot|_{U_i} \otimes A, d+l_i) \to (\mathcal{E}^\cdot|_{U_i} \otimes A, d+l_i')$ where $\{a_i\} \in \prod_i \operatorname{End}^0(\mathcal{E}^\cdot)(U_i) \otimes m_A$.

· As before, compatibility with differentials give

$$l_i' = e^{a_i} * l_i \qquad \text{for all } i \in I.$$

· Finally, the local isomorphisms e^{a_i} lift a global isomorphism in cohomology. Therefore the diagram with the gluing morphisms commutes in cohomology:

$$\begin{tikzcd}
(\mathcal{E}^\cdot|_{U_{ij}} \otimes A, d+l_j|_{U_{ij}}) \arrow[r, "e^{m_{ij}}"] \arrow[d, "e^{a_j}|_{U_{ij}}"'] & (\mathcal{E}^\cdot|_{U_{ij}} \otimes A, d+l_i|_{U_{ij}}) \arrow[d, "e^{a_i}|_{U_{ij}}"] \\
(\mathcal{E}^\cdot|_{U_{ij}} \otimes A, d+l_j'|_{U_{ij}}) \arrow[r, "e^{m_{ij}'}"'] & (\mathcal{E}^\cdot|_{U_{ij}} \otimes A, d+l_i'|_{U_{ij}})
\end{tikzcd}$$

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As before, taking the logarithm, this relation becomes

$$-m_{ij} \star -a_i|_{U_{ij}} \star m_{ij}' \star a_j|_{U_{ij}} = [d + l_j|_{U_{ij}}, b_{ij}]$$

for some $\{b_{ij}\} \in \prod_{i<j} \mathcal{E}nd^{-1}(\mathcal{E}^\cdot)(U_{ij}) \otimes m_A$.

<u>Remark:</u> We have shown that $\mathrm{Def}_{\mathcal{F}}$ is controlled by a sheaf of Differential graded Lie algebras $\mathcal{L}^\cdot = \mathcal{E}nd^*(\mathcal{E}^\cdot)$, via the Čech complex

$$\prod_i \mathcal{L}^\cdot(U_i) \longrightarrow \prod_{i<j} \mathcal{L}^\cdot(U_{ij}) \longrightarrow \prod_{i<j<k} \mathcal{L}^\cdot(U_{ijk}) \longrightarrow \cdots$$

<u>Definition:</u> $\mathrm{Def}_{\mathcal{L}^\cdot, \mathcal{U}}$ where

$$\mathrm{Def}_{\mathcal{L}^\cdot,\mathcal{U}}(A) = \left\{ (l,m) \in \left(\prod_i \mathcal{L}^1(U_i) \oplus \prod_{i<j} \mathcal{L}^0(U_{ij})\right) \otimes m_A \;\middle|\; \begin{array}{l} dl_i + \frac{1}{2}[l_i, l_i] = 0 \\ l_i|_{U_{ij}} = e^{m_{ij}} * l_j|_{U_{ij}} \\ m_{jk} \star -m_{ik} \star m_{ij} = \dots \end{array} \right\} \Big/ \sim$$

where $(l,m) \sim (l', m')$ if and only if there exists $(a,b) \in \left(\prod_i \mathcal{L}^0(U_i) \oplus \prod_{i<j} \mathcal{L}^{-1}(U_{ij})\right) \otimes m_A$ such that

$$\begin{cases} l_i' = e^{a_i} * l_i \\ -m_{ij} \star -a_i|_{U_{ij}} \star m_{ij}' \star a_j|_{U_{ij}} = [d + l_j|_{U_{ij}}, b_{ij}]. \end{cases}$$

<u>Theorem</u> [Fiorenza – Iacono – Martinengo]

The limit over open covers is a well defined functor

$$\mathrm{Def}_{\mathcal{L}} = \varinjlim_{\mathcal{U}} \mathrm{Def}_{\mathcal{L},\mathcal{U}}$$

There exists a natural isomorphism of deformation functors $\mathrm{Def}_{\mathcal{F}} \cong \mathrm{Def}_{\mathcal{L}}$, where $\mathcal{L}^\cdot = \mathcal{E}nd^*(\mathcal{E}^\cdot)$.